IV. "A Class of Functional Invariants." By A. R. FORSYTH, M.A., F.R.S., Fellow of Trinity College, Cambridge. Received March 7, 1888.

(Abstract.)

The memoir is occupied with the investigation of a class of functional invariants, constituted by combinations of the partial differential coefficients of a function of more than one independent variable. As the number of independent variables is limited to two, partly for the sake of conciseness, the general definition of such an invariant is that it is *a function ϕ of the partial differential coefficients of a dependent variable* z *with regard to* x *and* y, *such that when the independent variables are changed to* X *and* Y, *and the same function* Φ *is formed with regard to the new variables, the relation*

$$\Phi = \mathrm{J}^m \phi$$

is satisfied, where

$$\mathrm{J} = \frac{\partial(x, y)}{\partial(\mathrm{X}, \mathrm{Y})}$$

The transformations for which detailed results are given are of the homographic types:

$$\frac{x}{\alpha_1+\beta_1\mathrm{X}+\gamma_1\mathrm{Y}} = \frac{y}{\alpha_2+\beta_2\mathrm{X}+\gamma_2\mathrm{Y}} = \frac{1}{\alpha_3+\beta_3\mathrm{X}+\gamma_2\mathrm{Y}^3},$$

so that

$$\mathrm{J} = \begin{vmatrix} \alpha_1 , & \alpha_2 , & \alpha_3 \\ \beta_1 , & \beta_2, & \beta_3 \\ \gamma_1 , & \gamma_2 , & \gamma_3 \end{vmatrix} (\alpha_3+\beta_3\mathrm{X}+\gamma_3\mathrm{Y})^{-3}.$$

The characteristic properties of such invariants are—

(i.) Every invariant is explicitly free from the variables z, x, y, but necessarily contains p and q;

(ii.) It is homogeneous in the differential coefficients of z, and is of uniform and the same grade in differentiations with regard to each of the independent variables;

(iii.) It is symmetric or skew symmetric with regard to these differentiations;

(iv.) It satisfies four form-equations, viz.:—

$$\Delta_1\phi = q\frac{\partial\phi}{\partial p} + 2p\frac{\partial\phi}{\partial r} + 2\left(t\frac{\partial\phi}{\partial c} + 2s\frac{\partial\phi}{\partial b} + 3r\frac{\partial\phi}{\partial a}\right) + \ldots\ldots = 0,$$

$$\Delta_2\phi = p\frac{\partial\phi}{\partial q} + 2q\frac{\partial\phi}{\partial t} + 2\left(r\frac{\partial\phi}{\partial b} + 2s\frac{\partial\phi}{\partial c} + 3t\frac{\partial\phi}{\partial d}\right) + \ldots\ldots = 0,$$

$$\Delta_3\phi = q\frac{\partial\phi}{\partial s} + 2s\frac{\partial\phi}{\partial r} + t\frac{\partial\phi}{\partial s} + 3b\frac{\partial\phi}{\partial a} + 2c\frac{\partial\phi}{\partial b} + c\frac{\partial\phi}{\partial d} + \ldots = 0,$$

$$\Delta_4\phi = p\frac{\partial\phi}{\partial s} + 2s\frac{\partial\phi}{\partial t} + r\frac{\partial\phi}{\partial s} + 3c\frac{\partial\phi}{\partial d} + 2b\frac{\partial\phi}{\partial c} + a\frac{\partial\phi}{\partial b} + \ldots = 0,$$

and two index-equations, viz.:—

$$3\lambda\phi = 2p\frac{\partial\phi}{\partial p} + q\frac{\partial\phi}{\partial q} + 4r\frac{\partial\phi}{\partial r} + 3s\frac{\partial\phi}{\partial s} + 2t\frac{\partial\phi}{\partial t} + \ldots\ldots$$

$$3\lambda\phi = p\frac{\partial\phi}{\partial p} + 2q\frac{\partial\phi}{\partial q} + 2r\frac{\partial\phi}{\partial r} + 3s\frac{\partial\phi}{\partial s} + 4t\frac{\partial\phi}{\partial t} + \ldots\ldots,$$

in the last two of which λ is the index, an integer determinable from the form of ϕ by inspection. In these equations p and q are the partial differential coefficients of the first order; r, s, t those of the second order; a, b, c, d those of the third order; and so on.

An invariant is said to be *proper* to the rank n, when the highest differential coefficient of z occurring in it is of order n. By means of the solutions of the equations $\Delta_1\phi = \Delta_2\phi = \Delta_3\phi = \Delta_4\phi = 0$, considered as simultaneous partial equations, and by using the remaining equations, the following propositions relating to irreducible invariants in a single dependent variable z are established:—

Invariants can be ranged in sets, each set being proper to a particular rank;

There is no invariant proper to the rank 1, and there is one, viz., $q^2r - 2pqs + p^2t$, proper to the rank 2;

There are three invariants proper to the rank 3;

For every value of n greater than 3, there are $n+1$ invariants proper to the rank n, which can be chosen so as to be linear in the partial differential coefficients of order n.

Every invariant can be expressed in terms of this aggregate of irreducible invariants; and the expression involves invariants of rank no higher than the order of the highest differential coefficient which occurs in that invariant.

A special class of invariants, proper to ranks in numerical succession, is given by combinations of A_0, A_1, A_2, ... where—

$$A_0 = (r, s, t\nabla q, -p)^2,$$

$$A_1 = (a, b, c, d\nabla q, -p)^3,$$

$$A_2 = (e, f, k, h, i\nabla q, -p)^4, \ldots\ldots;$$

and the combinations are such that, when the transformations

$$A_{m-2} = m\,!\,(m-1)\,C_{m-2}$$

are effected, the resulting forms are the same combinations of the quantities C as the leading coefficients of the fundamental covariants of a binary quantic.

Some of the properties of the irreducible invariants involving differential coefficients of two dependent variables z and z' are obtained; and in particular it is shown that there is a single simultaneous irreducible invariant, $pq'-p'q$ proper to the rank 1, and that there are four such invariants proper to the rank 2.

The theory of eduction is next considered. A number of eductive operators similar to $A_0^{-\frac{1}{2}}\left(q\frac{\partial}{\partial x}-p\frac{\partial}{\partial y}\right)$ are given; such an operator, applied to an absolute invariant, educes another absolute invariant. Some illustrations are given, and some results, the analogues of reversor operations, are obtained by means of successive educts.

Finally it is shown that the theory of binary forms can be partly connected with the theory of functional invariants. The equations $\Delta_3\phi = 0 = \Delta_4\phi$ are satisfied by A_0, A_1, A_2, . . ., so that these quantities may be regarded as a succession of binary quantics in q and $-p$ as variables; and the same equations are characteristic of the simultaneous concomitants of such quantics. The functional invariants can therefore be expressed in terms of these simultaneous concomitants.

Presents, March 15, 1888.

Transactions.

Batavia:—Bataviaasch Genootschap van Kunsten en Wetenschappen. Tijdschrift voor Indische Taal-, Land- en Volkenkunde. Deel XXXII. Aflev. 2. 8vo. *Batavia* 1888. The Society.

Cambridge, Mass.:—Harvard College. Museum of Comparative Zoology. Bulletin. Vol. XIII. No. 6. 8vo. *Cambridge* 1887. The Museum.

Dublin:—Royal Irish Academy. Proceedings (Polite Literature and Antiquities). Vol. II. No. 8. Ditto (Science). Vol. IV. No. 6. 8vo. *Dublin* 1888; Transactions. Vol. XXIX. Parts 1–2. 4to. *Dublin* 1887; "Cunningham Memoirs." No. 4. 4to.